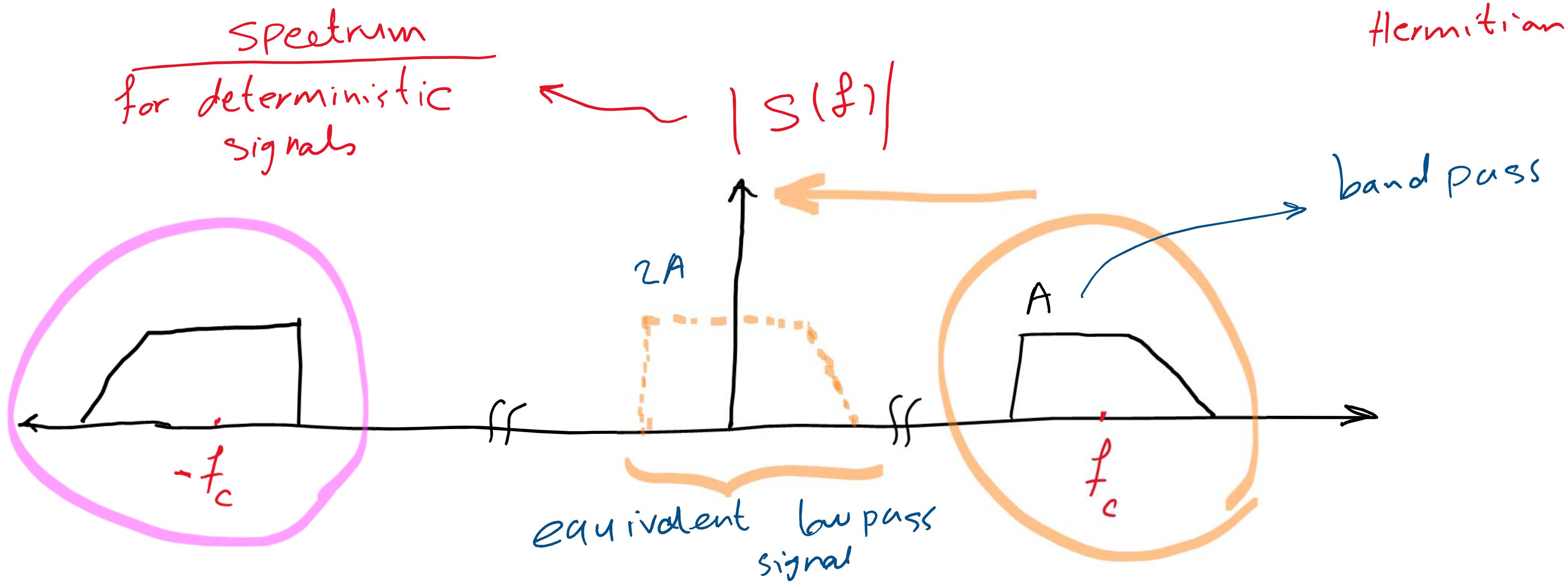
In the name of GOD

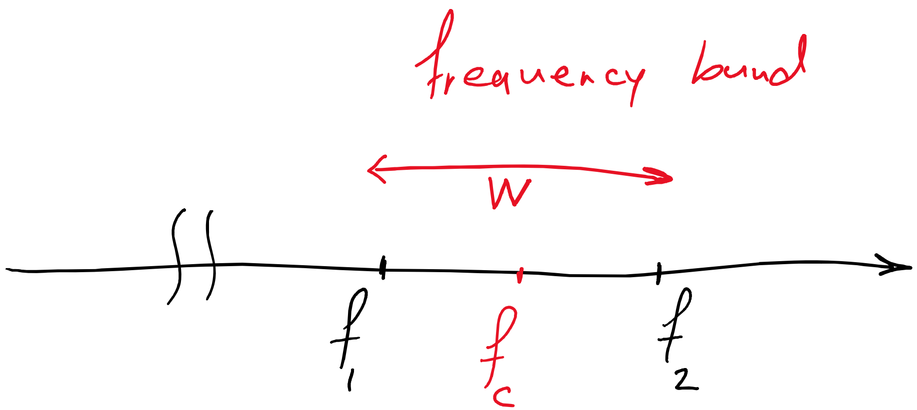
Reperesntation of high frequency signals based on equivalent lowpass signal

As you may know, digital Modulation is an interface between the digital blocks of a Communication System and physical Communiation channel. In the Communication system, we have a definite frequency range to transmit the source information to destination. So, the digital modulation should prepare a proper signal based on the channel characteristics and in the proper frequency range. So, the modulated signals are usually high frequency signals. In this session we want to find a representation of high freq. signals based on their equivalent low_pass signals. Because the mathematical analysis of Comm. signals is easier in the low frequencies.

we will see that the freq. range of Comm. systems does not impact on the error performance of the system. So, our performance analysis is done in low-freq. range.

As you may know, any real bandpass signal, $s(t)$, has a furier transform, $S(f)$, that has the Hermitian Symmetry means $|S(f)|$ is an even function of f and $\angle S(f)$ is an odd function of f and we have $\underbrace{S^*(f) = S(-f)}_{\text{Hermitian symmetry}}$





W : frequency bandwidth

if $W \ll f_c$: narrow band

central frequency

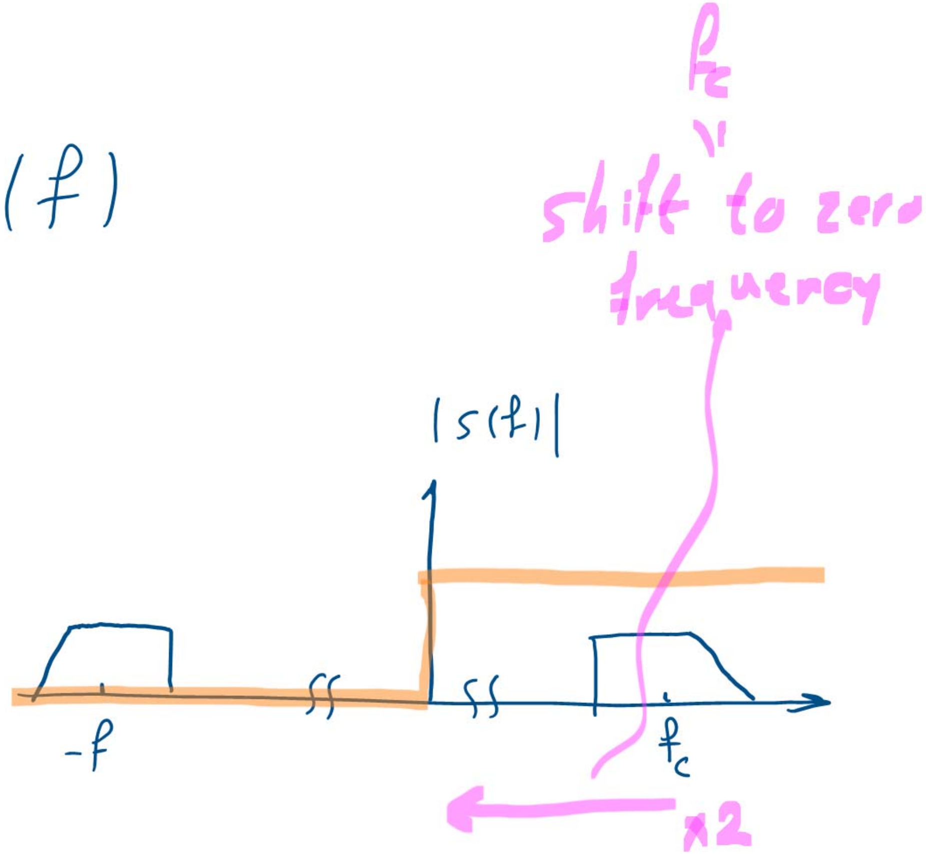
or carrier frequency

if the bandwidth of a system (signal) is so smaller than the central frequency, we call it narrow-band system (signal)

In this course we consider narrow-band systems (signals)

Now, we want to find how we can represent a narrow-band band-pass deterministic signal, $s(t)$, based on its equivalent low pass signal. As we in the figure, we can write

$$\underbrace{S_+(f)}_{\text{part of } s(f) \text{ in positive freq.}} = 2\,\underbrace{U(f)}_{\text{Unit step function in frequency domain}}\, S(f)$$



$$\Longrightarrow \quad \underbrace{S_\ell(f)}_{\text{equivalent lowpass signal of } s(t) \text{ in freq. domain}} = \underbrace{S_+(f+f_c)}_{\text{shift to zero}}$$

It is the operation we sould do, in frequency domain to find the equivalent low-pass signal. But we should also find a representation of bandpass signal based on equivalent low pass signal in time domain. So, we use inverse furier transform to find this representation. we know that

$$s_+(t) = FT^{-1}\{S_+(f)\}$$

$$= FT^{-1}\{2\,U(f)\,S(f)\}$$

$$= 2\,\underbrace{FT^{-1}\{U(f)\}}_{\frac{1}{2}\delta(t) + \frac{j}{2\pi t}} * \underbrace{FT^{-1}\{S(f)\}}_{s(t)}$$

furier Trans. characterstics

FT from table

$$\Rightarrow s_+(t) = 2\left(\frac{1}{2}\delta(t) + \frac{j}{2\pi t}\right) * s(t)$$

$$\Rightarrow s_+(t) = s(t) + \underbrace{j\frac{1}{\pi t} * s(t)}_{\hat{s}(t)} \Longrightarrow \boxed{s(t) = Re\{s_+(t)\}}$$

$\hat{s}(t)$ is the Hillbert Transform of $s(t)$, means the filteration of $s(t)$ by a filter by $\frac{1}{\pi t}$ impulse response.

As a summary

$$\underbrace{S_\ell(f)}_{\text{equivalent}} = S_+(f+f_c) \quad (\mathrm{I}) \qquad \boxed{\Longrightarrow \; S_\ell(t) = S_+(t)\, e^{-j2\pi f_c t}}$$

equivalent lowpass signal in frequency domain

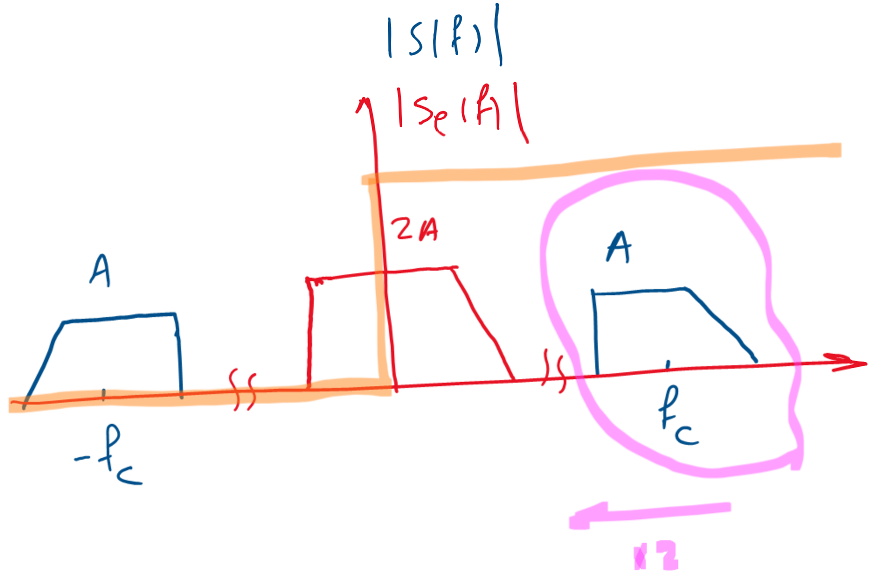
$$s(t) = \mathrm{Re}\{S_+(t)\} \quad (\mathrm{II})$$

$$(\mathrm{I}),(\mathrm{II}) \Longrightarrow \underbrace{s(t)}_{\substack{\text{band-pass}\\ \text{signal}}} = \mathrm{Re}\{\underbrace{S_\ell(t)}_{\substack{\text{equivalent}\\ \text{Low-pass}\\ \text{signal}}} e^{j2\pi f_c t}\}$$

equivalent Low-pass signal / Complex envelope of $s(t)$

As you know $s_\ell(t)$ is a Complex signal (show this as an excercise) (send it with assignment #1)

So, we can rewrite $s_\ell(t)$, as



$$s_\ell(t) = \underbrace{Re\{s_\ell(t)\}}_{x_\ell(t)} + j\,\underbrace{Im\{s_\ell(t)\}}_{y_\ell(t)}$$

$$\Longrightarrow \quad s_\ell(t) = x_\ell(t) + j\,y_\ell(t)$$

$$s(t) = \operatorname{Re}\{ s_\ell(t)\, e^{j2\pi f_c t} \} \quad (1)$$

$$s_\ell(t) = x_\ell(t) + j\, y_\ell(t) \quad (2)$$

(1), (2) $\Longrightarrow$

$$s(t) = \operatorname{Re}\{ (x_\ell(t) + j\, y_\ell(t))\, \underbrace{e^{j2\pi f_c t}}_{(\cos 2\pi f_c t + j \sin 2\pi f_c t)} \}$$

$$\Rightarrow s(t) = x_\ell(t)\,\underbrace{\cos 2\pi f_c t}\; - \;y_\ell(t)\,\underbrace{\sin 2\pi f_c t}$$

orthogonal signals

$$s_\ell(t) = x_\ell(t) + j\,y_\ell(t)$$

Quadrature elements of $s(t)$

$$s(t) = \underbrace{x_\ell(t)}_{\text{Inphase}}\,\cos 2\pi f_c t \; - \; \underbrace{y_\ell(t)}_{\text{Quadrature}}\,\sin 2\pi f_c t$$

Inphase — I

Quadrature — Q

This representation show that $s(t)$ include, the real and imaginary part of $s_\ell(t)$, modulated on two orthogonal signal (Cos, sin). So, $\underbrace{x_\ell(t)}_{\text{real part of } s_\ell(t)}$ and $\underbrace{y_\ell(t)}_{\text{Im. part of } s_\ell(t)}$ are called the

Quadrature Elements of $s(t)$.

1) $\underbrace{s(t) = Re\left\{ s_\ell(t)\, e^{j2\pi f_c t} \right\}}_{\text{representation } \# 1}$

2) $\underbrace{s(t) = x_\ell(t) \cos 2\pi f_c t - y_\ell(t) \sin 2\pi f_c t}_{\text{rep. } \# 2}$

$s_\ell(t) = \underbrace{x_\ell(t)}_{Re\{s_\ell(t)\}} + j\, \underbrace{y_\ell(t)}_{Im\{s_\ell(t)\}}$

As you know, every Complex signal Can be written by the use of Amplitude and phase of signal, so, we Can rewrite $s_e(t)$ as follows

$$s_\ell(t) = x_\ell(t) + j\,y_\ell(t) = |s_\ell(t)|\, e^{j\angle s_\ell(t)}$$

in which

$$\underbrace{|s_\ell(t)|}_{a(t)} = \sqrt{x_e^2(t) + y_e^2(t)} \quad \& \quad \underbrace{\angle s_\ell(t)}_{\theta(t)} = \tan^{-1}\frac{y_e(t)}{x_e(t)}$$

$$\Rightarrow \quad s_\ell(t) = a(t)\, e^{j\theta(t)}$$

in which

$$\underbrace{a(t)} = \sqrt{x_\ell^2(t) + y_\ell^2(t)} \quad , \quad \underbrace{\theta(t)} = \tan^{-1} \frac{y_\ell(t)}{x_\ell(t)}$$

<u>Amplitude</u> of $s(t)$ — phase of $s(t)$

envelope

In other hand we know

$$s(t) = Re\left\{ s_l(t)\, e^{j2\pi f_c t} \right\}$$

$$s(t) = Re\left\{ a(t)\, e^{j\theta(t)}\, e^{j2\pi f_c t} \right\}$$

$$s_l(t) = a(t)\, e^{j\theta(t)}$$

$$s(t) = a(t) \cos\left(2\pi f_c t + \theta(t)\right)$$ rep # 3

As a summary

1) $$s(t) = \mathrm{Re}\left\{ s_\ell(t)\, e^{j2\pi f_c t} \right\}$$

2) $$s(t) = x_\ell(t) \cos 2\pi f_c t - y_\ell(t) \sin 2\pi f_c t$$

$$\left(s_\ell(t) = x_\ell(t) + j\, y_\ell(t) \right)$$

3) $$s(t) = a(t) \cos\left(2\pi f_c t + \theta(t) \right)$$

$$\left(s_\ell(t) = a(t)\, e^{j\theta(t)} \right)$$

Now, we want to find an expression for band-pass signal Energy based on the energy of equivalent low-pass signal. As you know

$$\left(\text{Energy of } s(t)\right) = E_s = \int_{-\infty}^{\infty} s^2(t)\, dt = \int_{-\infty}^{\infty} a^2(t) \underbrace{\cos^2(2\pi f_c t + \theta(t))}_{\frac{1}{2}\left[1+\cos(4\pi f_c t + 2\theta(t))\right]} dt$$

$$\int_{-\infty}^{\infty} |s(t)|^2\, dt$$

s(t) is real signal

rep #3

$$\Rightarrow E_s = \underbrace{\frac{1}{2}\int_{-\infty}^{\infty} a^2(t)\,dt}_{\underbrace{\int_{-\infty}^{\infty} |s_\ell(t)|^2\,dt}} + \underbrace{\frac{1}{2}\int_{-\infty}^{\infty} a^2(t)\, C\left(4\pi f_c t + 2\theta(t)\right) dt}_{=0}$$

E_{s_ℓ} = Energy of $s_\ell(t)$

$$\Longrightarrow \underbrace{E_s}_{\text{Energy of band-pass signal}} = \frac{1}{2}\, \overbrace{E_{s_\ell}}$$

→ Energy of equivalent low-pass signal